Modeling of the Mechanics of Fiber—Reinforced Three-D Printed Parts based on Continuous—Loop Reinforcement pioneered by Markforged™.

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Abstract. *This is a very short white paper that takes a look at the modeling of the mechanics of fiber-reinforced, three-D printed parts such as those produced by Markforged™ printers. This brief paper analyzes only bending and axial loading of a very simple continuous fiber-reinforced beam.*

I. Introduction

Continuous-loop fiber reinforcement is a technique used to significantly increase the strength of three-D printed parts by inlaying continuous strands of a much stronger material within the infill of a three-D printed plastic. The technique was pioneered by Markforged™ and is used in their printers to produce three-D printed parts that have the same strength as aluminum—by Markforged's metrics [1].

Regardless, this paper was written to understand the mechanics of continuous-loop reinforcement and to propose a model for the mechanical modeling of continuous-loop reinforced parts using simple hand-calculations. We will create this model for both bending as well as axial loading.

It's important to note, with continuous-loop reinforcement, that we need to pay careful attention to the direction of the fibers to get the most out of the reinforcement. But this will be discussed in the next section. At the end we will analyze how correct our model and assumptions are.

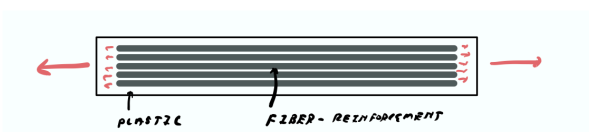
II. Overview of Continuous-Loop Reinforcement from [2]

The best overview of how continuous-loop reinforcement works comes from the YouTube video linked in [2]. We recommend the reader watch this video, but a short summary is included in this paper as well.

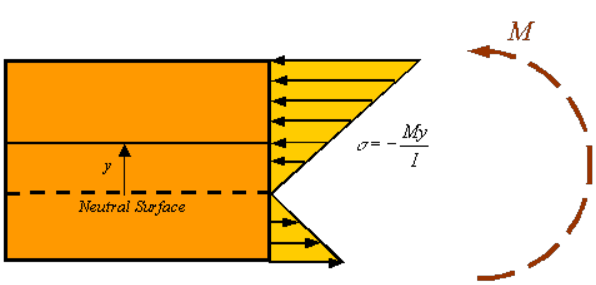
II.1.0 Fibers + Fiber Strength. Before we get into print reinforcement, let's talk about the fibers. We'll define a fiber, simply, as a thin strand of a high-strength material such as carbon-fiber, fiberglass, and etc [5]. Like most fibers, these fibers are incredibly strong in *tension,* provide little resistance in *compression,* and do not have a defined bending stiffness. You can think of these fibers as being very similar to fishing line.

II.2.0 Fibers + Three-D Printed Parts. Three-D printers can take advantage of the properties of these fibers by strategically inlaying them into a part. The direction the fibers are laid is up to the engineer who will likely choose to lay them in a direction that takes advantage of their incredible tensile strength. An example of this is if we simply have a beam under a tensile load, we would inlay the fibers axially so we may take advantage of their tensile strength.*

**Note that since these fibers do not provide much compressive resistance, if this beam were in compression the fibers would do little to reinforce the part.*



Pure bending also takes advantage of the fiber's tensile strength as pure bending produces axial stresses in the beam.



We will analyze pure bending as well as tensile loading in this paper. For more complicate modeling and torsion, it is likely best to consult Markforged™ directly. The purpose of this paper is to present methods that could be used to create a more accurate order-of-magnitude estimate for design purposes.

III. Modeling of Continuous-Loop Reinforcement

To create a simple model of continuous loop reinforced parts we will make the following assumptions.

- The fibers provide no resistance to any loading other than in the purely tensile direction.
- Stress is taken only by the plastic in any part of the three-D print where the fibers do not feel tension.
- Tensile stress is taken fully by the fibers.
- The fibers have been laid in the axial direction along the beam.

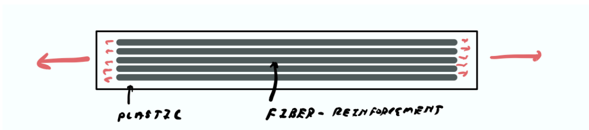
Let's set the following variable definitions for the rest of the paper.

Variables:
σ = Axial Stress (N/m^2)
A_1 = Cross Sectional Area of Fibers (m^2)
$2h$ = Beam Height (m)
w = Beam Thickness (m)
A_2 = Cross Sectional Area of Plastic (m^2)
E_1 = Young's Modulus of Fibers (N/m^2)
E_2 = Young's Modulus of Plastic (N/m^2)
n = Distance to Neutral Axis (m)
M = Pure Moment Load (Nm)
F = Pure Force Load (N)

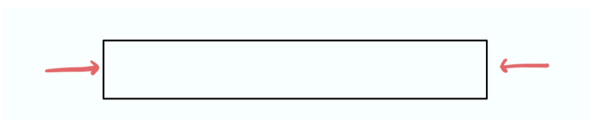
It is also important to note that this paper won't present any calculations on deflection, stress, strain, and etc. This paper will only discuss how to model the material which can be used for hand calculations in future work.

III.1.0 Pure Axial Loading of Reinforced Parts. As we know, pure axial loading comes in two forms pure tension, and pure compression. The discussion of axial loading will be brief as most of the information was presented in the assumptions section. We will simply formalize the assumptions regarding axial loading and diagram them in this section. The section regarding bending will be more complex.

If the reinforced part is in tension and the fibers have been laid in the axial direction, we end up with the same diagram seen on the previous page (reproduced below).



Since the fibers take most of the axial load, we can model this composite beam as *not* a composite but a regular elastic beam with a Young's Modulus E_1 equal to that of the Young's Modulus of the fibers.



In compression—however—we've stated that the fibers essentially provide no resistance in compression hence we can model the structure as a regular elastic beam but this time with a Young's Modulus E_2 of the plastic and not of the fibers. Below are typical values for E_1 and E_2.

Material	E (young's modulus)	Y (yield stress)
ABS Plastic	1.19 - 2.9 GPa	29.6 - 48 MPa
Carbon Fiber	228 GPa	4.62 - 3220 MPa

From the numerical values in the table and the simplistic modeling, we can likely assume that reinforcement will increase at least the axial stiffness of the produced part.

III.2.0 Pure Bending of Reinforced Parts. Bending is a little more complicated than axial loading even through in pure bending the stresses are only axial. It is recommended before continuing this paper the reader reviews simple axial bending as it will not be explained here.

https://ocw.mit.edu/courses/mechanical-engineering/2-001-mechanics-materials-i-fall-2006/lecture-notes/lec18.pdf

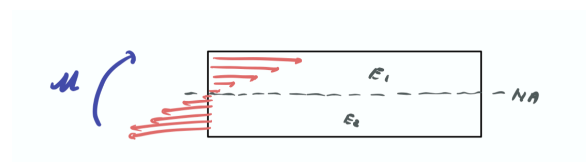
III.2.1 Stress Discussion + Materials. In bending it is known that part of the beam is in tension and part of the beam is in compression just like in the diagram on the previous page from [6]. It is also know that bending stress is defined as the following.

$$\sigma(x, y) = -y\frac{M(x)}{EI(x)}$$

Where "I" is the area moment of inertia of the beam's cross section and "y" is the vertical position from the beam's neutral axis.

From simple mechanics knowledge, the neutral axis of a beam for a non-composite, uniform beam would be the equal areas point of its cross-section. And further—as noted in [3]—an asymmetrical beam made of composite materials does not necessarily have its neutral axis at the center of the beam. Now the reason for this will not be explained in detail here, but the short answer is that the strain at any point along the beam must be *continuous in the y-direction*. If it is not, the beam has obviously broken. Due to this geometric compatibility constraint, an asymmetrical beam or a composite beam will see a shift in its neutral axis from center. We will discuss the nature of this shift later but we say this to make the point that we cannot assume that the neutral axis will be at the center of the beam.

Our beam in pure bending essentially is made of a material which has a young's modulus E_1 in tension and E_2 in compression. This would look something like the diagram below.

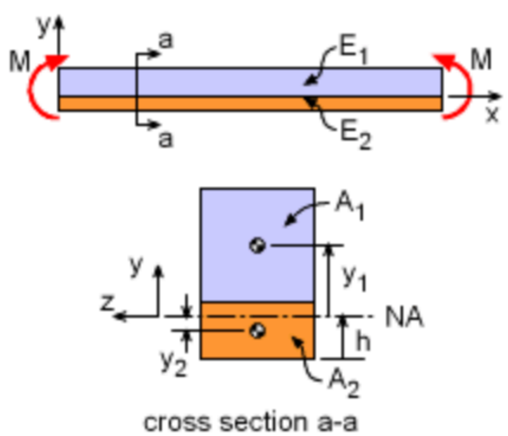


So how do we model something like that? Well the first thing we'd need to determine is the direction of curvature. For example, will this beam bend up or down? Knowing this will allow us to identify which part of the beam is in tension and which part of the beam is in compression. If the loading is more complicated, you will need to do this for every part of the beam where the direction of the bending changes.

- Step one, figure out the direction of bending to determine if the top or bottom of the beam is in tension or compression.

All parts of the beam that are in tension will have a young's modulus = E_1 and the parts in compression will have a young's modulus = E_2.

III.2.2 Determining the location of the Neutral Axis. In bending, to know which parts of the beam are in tension and compression we need to determine the location of the neutral axis. As noted, one side of this axis will be in tension and the other side in compression. The location of the neutral axis comes purely from geometric compatibility constraints as discussed in III.2.1.



Neutral Axis Location for Composite Beam

As discussed in [3], for a composite beam we're looking for the point where the EIeff above the neutral axis is the same as the EIeff below the neutral axis. For our material this is actually simpler since no matter what the location of the neutral axis is the material on either side will be uniform and one of two different young's moduli E_1 or E_2.

The two integrals are the first moment of each material area which is commonly noted as simply Q, giving

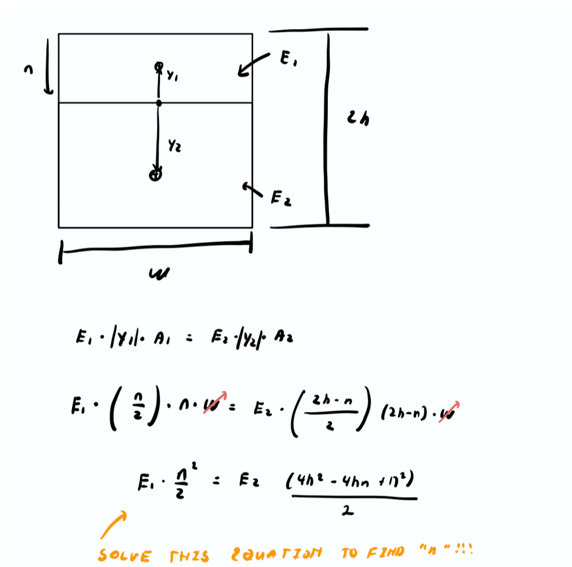
$$0 = E_1Q_1 + E_2Q_2$$

Generally, Q is not solved using the integral form since the centroid of each material area will be known (or found in the Sections appendix). Thus the equation can also be written as

$$0 = E_1 (y_1 A_1) + E_2 (y_2 A_2)$$

where y_1 and y_2 are the distance from the NA to the centroid of the material area. Notice, "h" is not in this equation, but both y_1 and y_2 depend on h. Thus, the only unknown will be h and can be determined. Note, y will be negative if the centroid of the material area is below the NA.

The above is also from [3] and it is what we will use to solve for the location of the neural axis. We will perform this calculation for a square cross section.



Once we've found the neutral axis we can then perform the calculation as we normally would.

- Calculate the EI_eff(x) of the cross section.
- Apply the stress formula to find the stress.
- Find the axial strain by multiplying by the correct young's modulus (E_1 or E_2).

IV. Analysis

This final section will simply present the validity of our assumptions and some points of caution for utilizing this method to model fiber reinforced three-D printed parts.*

**Note that this method only works for continuous loop reinforcement not chopped strand reinforcement.*

IV.1.0 Assumptions Regarding Stress. First, the statement that tensile stress is taken fully by the fibers is not accurate. It could be accurate to say compression is taken only by the plastic but in tension both the plastic and the fibers share the tensile force. It's an entirely new composite material with some young's modulus E_3 which is a combination of that of the plastic and the carbon fiber in tension. It's possible that E_3 could be calculated by determining the EA_effective of the plastic and the carbon fiber in the tensile region, or the interaction could be much more complicated.

We also assumed that the fibers only provide a tensile resistance to stress and this may be true of the fibers on their own. But it's also possible that when encased in the plastic the fibers can increase quantities like shear strength of the part but these interactions were not modeled or discussed as they are much more complicated.

IV.2.0 Assumptions Regarding Laying of Fibers. It's also incredibly important to note that any derivations above assume that the fibers have been laid in the axial direction. However, according to [1] the direction in which the part is printed means—for example—if we wanted to increase the torsional stiffness of the beam we could do so by inlaying the fibers in a different direction. These models are not presented in this paper but information from this paper may be utilized to derive such a model.

IV.3.0 Geometric Assumptions. Finally, note that the simplification presented in II.2.2 that lead to no integration in the calculation is valid when "y1" or "y2" is the distance to the centroid of the area above or below the neutral axis. Our calculation is only valid for a square cross section but —using [7]—may be adapted to different cross sections.

V. Conclusions

Overall, this paper presents some basic information about the basics of fiber reinforcement and how one can think about modeling the mechanics of a fiber-reinforced three-D printed part. It's important to note that based on the assumptions the approach outlined here is likely good for order-of-magnitude estimates in design calculations and likely should not be used for more rigorous applications. For these, it's advisable to contact the experts in fiber-reinforced parts—Markforged™.

VI. References

[1] *Markforged Continuous Fiber Printing:* https://markforged.com/materials/

[2] *Continuous v/ Chopped Reinforcement for CF:* www.youtube.com/watch?v=0pDNHqNzwNU

[3] *Mechanics of Composite Materials:* http://www.ecourses.ou.edu/cgi-bin/ebook.cgi?topic=me&chap_sec=06.1&page=theory

[4] *MIT 2.001 Course:* https://ocw.mit.edu/courses/mechanical-engineering/2-001-mechanics-materials-i-fall-2006/

[5] *CF Roll Picture:* https://www.mark3d.com/en/product/filaments-for-markforged-3d-printers/markforged-carbon-fibre-cff-filament-roll-volume-upon-selection/

[6] *Pure Bending Stresses:* http://emweb.unl.edu/NEGAHBAN/Em325/11-Bending/Bending.htm

[7] *Differing Cross Sections:* http://www.ecourses.ou.edu/cgi-bin/eBook.cgi?doc=session_blank&topic=me&chap_sec=&page=&appendix=sections

Appendix A. Example Design of Fiber—Reinforced Part under a Compressive Load

A.1.0 Short Explanation. This appendix presents an initial sketch of what the design of a fiber-reinforced three-D printed part could look like if we were designing for compression, and how to inlay the fibers.