2.001 Mechanics & Materials: Project Three

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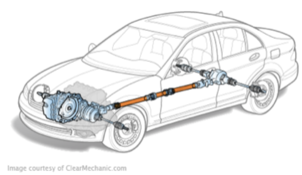
1 Selecting a Situation

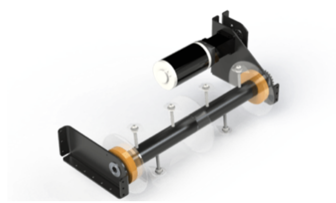



Figure 1: (In Order) Laptop, Car Drive Shaft, Torsion Spring, Rope Climber, Sawhorse Table

- **Laptop Screen Bending:** The first situation I thought of was a laptop screen when it gets pushed all the way to its most open position. What forces would I need to exert to yield the laptop screen if I kept pushing. *I chose not to analyze this as this is not bending of a slender beam, but a plate.*
- **Car Main Drive Shaft:** The main drive-shaft of a car transfers the torque between the engine and the rear wheels so I could analyze the torsion in this shaft. *I chose not to analyze this as it seemed very simple.*
- **Torsion Spring:** A torsion spring can be modeled as a beam in bending that's connected to a lateral bar in torsion (and bending). *I chose not to analyze this as it seemed complicated and I wasn't sure how to model the spring as a bar in torsion.*
- **Sawhorse Table:** I was curious as to why a home-made table made of 3/4" plywood was bending so much when heavy objects were placed on it and if it was getting close to failure. *I chose not to analyze this as this is not bending of a slender beam, but a plate.*
- **Rope Climber:** I chose to analyze the rope climber system so that is described in detail in the next section.

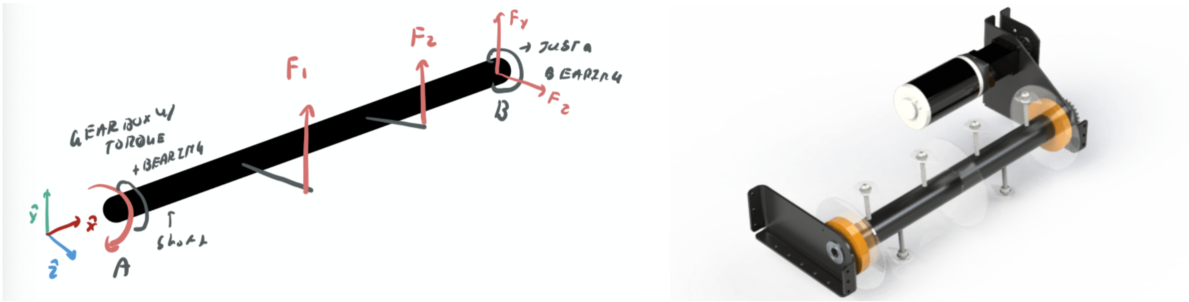


Figure 2: A simplification of the two-rope winch system.

2 Modeling & Abstraction

Figure 2 shows the system I have chosen to analyze which is a double-rope winch climber sometimes used in FIRST Robotics. A single shaft supported on either end by bearings and driven on one end by a motor wraps two ropes around its shaft to pull a robot up a wall. The actual picture in Figure 2 is of a single rope climber while the FBD shows the forces from two ropes, but it's easy to imagine two ropes being wound onto the shaft.

2.1 Assumptions

To translate this situation into a 2.001-style problem, we will make the following assumptions:

- **General:** We will assume for simplicity sake that the shaft is stationary at the moment of analysis and is not rotating. We will also assume that the torque provided by the motor is perfectly balanced by the torques provided by the ropes.
- **Loading Conditions:** We will also assume that the ropes are evenly spaced across the shaft and that they are being wrapped around pullies with a radius of 5cm. We will assume the bearings can only provide forces in the y and z direction and the system has no forces on it in the x direction as the ropes pull perfectly in the vertical direction. We can further assume that the weight of the robot is $600N$ and the weight is split evenly between the two ropes.
- **Zeros and Origin:** We will assume for simplicity sake that $\varphi(x = 0) = 0$ and that the displacements $\vec{u_A} = 0$ and $\vec{u_B} = 0$ where A and B are the locations of the bearings on the shaft. The location of $x = 0$ is at section A.

2.2 Diagram

Figure 3 summarizes all these assumptions and is located on the next page. The accuracy of these assumptions will be discussed in the analysis section.

3 Problem Statement

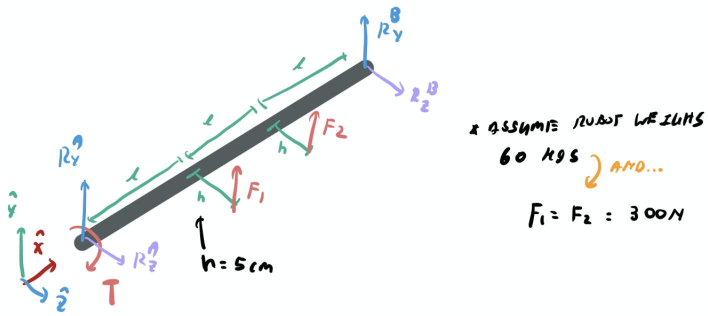


Figure 3: A rope-climber shaft in both torsion and bending.

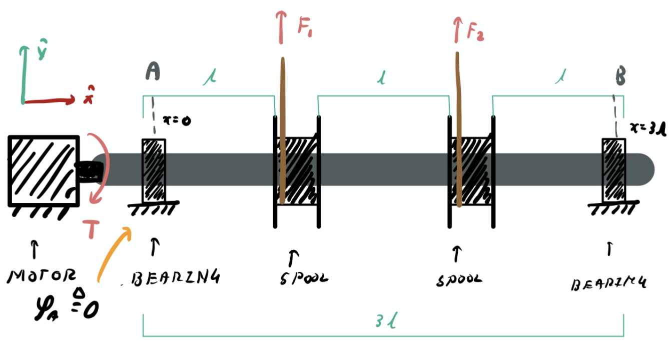


Figure 4: A rope-climber shaft in both torsion and bending.

The above shaft comes out of a dual-rope winch climber from a FIRST Robotics robot. The shaft has a length $3l$ equal to 0.6m, a diameter d of 1.5cm, and is supported by bearings at the endpoints A and B. A motor behind one of the bearings produces a torque T about the shaft to balance the torque produced by the rope forces $F_1 = F_2 = w = 300N$ which act at $h = 5cm$ from the axis of the shaft. The shaft has a uniform circular cross-section and is made of Aluminum with young's modulus $E = 70$ GPa, and shear modulus $G = 26$ GPa.

- Calculate the shear stress $\tau(x, r)$ resulting from torsion as a function of x and r in the bar. Which component of the stress tensor does $\tau(x, r)$ correspond to? What is the maximum shear stress and where does it occur?

- Calculate the axial stress $\sigma(x, y)$ resulting from bending as a function of x and y in the bar. Where does the maximum axial stress occur?

- Assuming the rope is inextensible, the spools are rigidly fixed to the shaft, and the rope does not slide around the spool, what is the upward displacement of the rope applying the force F_2 due to both torsional and bending deformation of the shaft. (You may use the Myosotis Tables for bending deformation).

- Without doing any calculations, identify the region of the shaft with the highest probability of failure based on stresses alone. *Hint: where are each of the stresses at a maximum?*

4 Calculations

4.1 Part I: Let's Start w/ Torsion

To find the stress $\tau(x, r)$ resulting from torsion as a function of x and r, we start by drawing an FBD of the system shown in Figure 5.

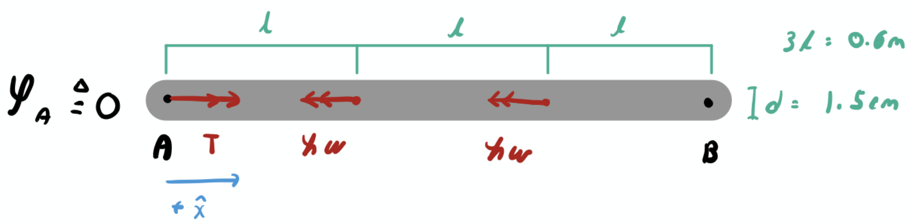


Figure 5: Free-Body-Diagram for torques acting on the shaft.

The first step to finding the stress is to find the internal torque resultant $\mathcal{T}(x)$. We do this by making strategic "cuts" in the bar and using rotational-equilibrium about the x axis. This is shown in Figure 6.

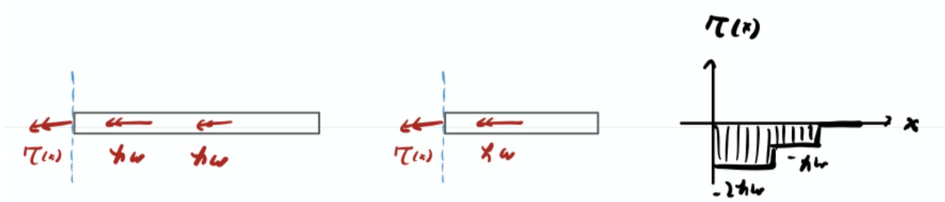


Figure 6: Free-Body-Diagram for $\mathcal{T}$ throughout the bar.

The above FBD and balancing the moments about the x axis leaves us with the following for $\mathcal{T}(x)$. For $w = 300N$, $h = 5cm$, and $hw = 15Nm$.

$$\mathcal{T}(x) = \begin{cases} -2hw & x \leq l \\ -hw & l < x \leq 2l \\ 0 & 2l < x \leq 3l \end{cases}$$

Now we can use the following equations (Equations 1) to solve for the stress $\tau(x, r)$ in the bar. Where $R = d/2 = 0.75$ cm.

$$\tau(x, r) = \frac{\mathcal{T}(x) * r}{IP_{eff}(x)} \quad IP_{eff}(x) = \pi/2 * R^4 \tag{1}$$

The left equation is the equation for shear stress as a function of x, r, the geometry and the internal resultant $\mathcal{T}(x)$. The right equation is the Polar MOI (Ip_{eff}) of a solid circular cross section. Now we can use these to calculate the stress as a function of x. Note that because r is on the top of the equation, the max stress will occur at the outermost part of the shaft from the center. We know that for this shaft that Ip_{eff} is constant across its length. Note that the units of $\tau(x, y)$ are MPa.

$$\tau(x, R) = \frac{2 * \mathcal{T}(x)}{\pi * (R)^3}$$

$$\tau(x, R) = \begin{cases} -45.3MPa & x \leq l \\ -22.6MPa & l < x \leq 2l \\ 0MPa & 2l < x \leq 3l \end{cases}$$

Now an important thing to note is that these torsional shear stresses are acting solely along the φ direction on and x-directed face. Therefore $\tau(x, y)$ corresponds to the $\sigma_{x\varphi}$ component of the stress tensor.

4.2 Part II: Onto Bending

To find the axial stresses from bending $\sigma(x, y)$ we start with the schematic in Figure 7.

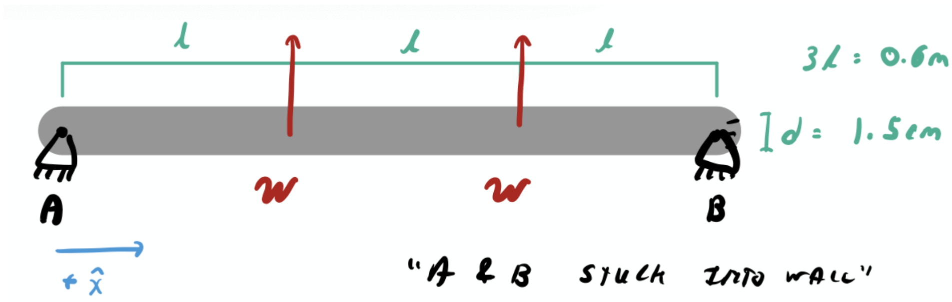


Figure 7: Schematic for bending loads acting on the shaft. We assume the bearings at A and B do not prevent the shaft from sloping.

The first step for us is to find the reaction forces $\mathcal{R}y_A$ and $\mathcal{R}y_B$ at the bearings which leads us to the FBD, below.

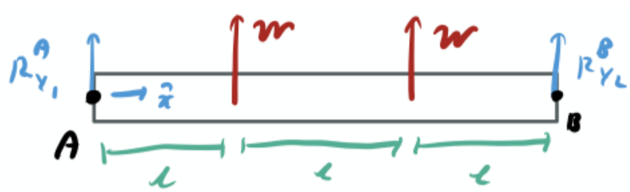


Figure 8: Free-Body-Diagram for bending. Includes reaction forces.

We can now use moment-equilibrium about A to find $\mathcal{R}y_B$ and then use the sum of the forces-$y = 0$ to find $\mathcal{R}y_A$.

$$\sum F_y = 0 \quad \sum M_{Z\,A} = 0 \tag{2}$$

$$\sum M_{Z\,A} = 0 \quad w * l + w * 2l + \mathcal{R}y_B * 3l = 0$$

$$\mathcal{R}y_B = -w = -300N$$

$$\sum F_y = 0 \quad \mathcal{R}y_A + w + w + \mathcal{R}y_B = 0$$

$$\mathcal{R}y_A = -w = -300N$$

Now we can use this information and the FBDs in Figure 8 to find the bending moment resultant throughout the shaft.

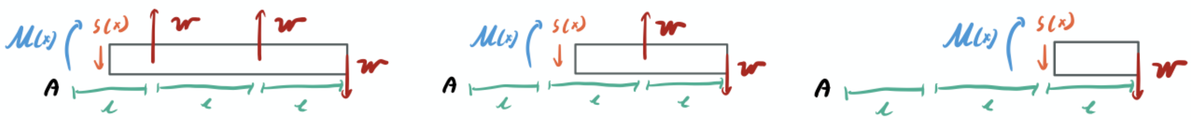


Figure 9: Bending moment FBDs through different sections of the bar.

We can now calculate the bending moment resultant $\mathcal{M}(x)$ throughout the bar. The units of $\mathcal{M}(x)$ are Nm.

$$\mathcal{M}(x) = \begin{cases} w*(l-x) + w*(2l-x) - w*(3l-x) = -w*x & x \leq l \\ w*(2l-x) - w*(3l-x) = -w*l & l < x \leq 2l \\ -w*(3l-x) & 2l < x \leq 3l \end{cases}$$

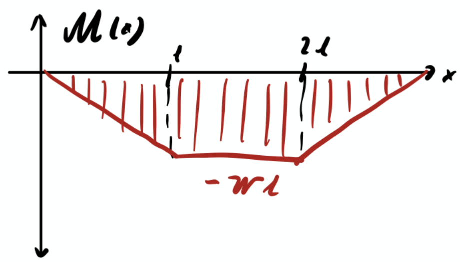


Figure 10: Graph of $M(x)$ through different sections of the bar.

We can now find the axial stress $\sigma(x,y)$. In this case y is the coordinate-y on the axis where $y = 0$ is at the mid-plane of the shaft. We use the following equations to do that where $\mathcal{I}(x)$ is the Second Moment of Area of a circular cross-section (not to be confused with the Polar MOI used in 4.1). We also know that for this specific bar that $\mathcal{I}(x)$ is a constant and $R = d/2 = 0.75$ cm.

$$\sigma(x,y) = -y * \frac{\mathcal{M}(x)}{\mathcal{I}(x)} \quad \mathcal{I}(x) = \pi * \frac{R^4}{4} \tag{3}$$

We also know that the maximum stress would occur where $|y|$ is the greatest. So that's. $y = \pm R$

$$|\sigma(x, \pm R)| = |\sigma(x)|_{max} = \frac{4 * M(x)}{\pi * R^3}$$

So the maximum axial stress from bending is in the region $l \leq x \leq 2l$ where:

$$|\sigma(x)|_{max} = \frac{4 * w * l}{\pi * R^3} = 181.08 MPa$$

4.3 Part III: Displacement Calculations

The next part asked us to determine the vertical displacement of rope #2 assuming the rope is not extensible, the pullies are rigidly attached to the shaft, and the rope does not slide on the pullies.

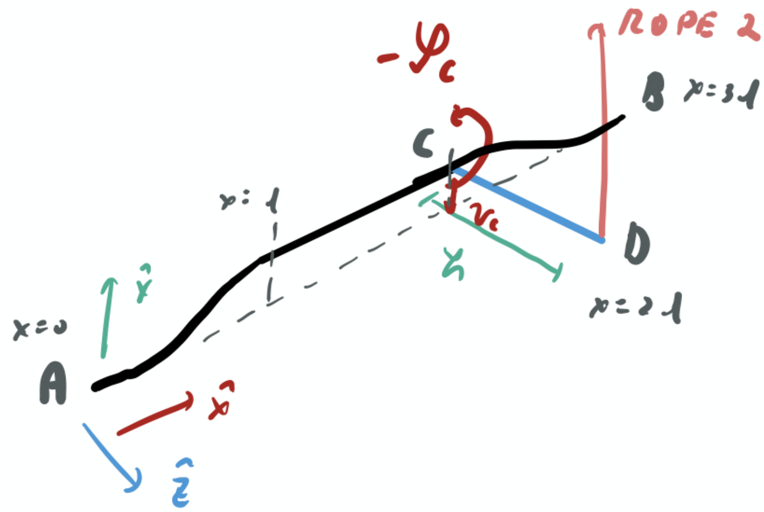


Figure 11: Components contributing to displacement of rope. Note that $-\varphi_c$ causes upwards movement of point D.

Looking at the diagram we can very simply say (with the assumptions we have made) that the vertical displacement of the rope v_d is the sum of the vertical displacement from bending at point C, v_c and the vertical displacement at D resulting from torsion $-\varphi_c * h$.

$$v_d = v_c - \varphi_c h$$

4.3.1 Calculating $\varphi_c h$

To calculate $\varphi_c h$ we need to remember that the following equations are true for a shaft in torsion.

$$\frac{d\varphi}{dx} = \frac{\mathcal{T}(x)}{G * Ip_{eff}(x)} \tag{4}$$

$$\mathcal{T}(x) = \begin{cases} -2hw & x \leq l \\ -hw & l < x \leq 2l \\ 0 & 2l < x \leq 3l \end{cases}$$

So now all we need to do is integrate this function from 0 to $2l$ to get φ_c. Note that $Ip_{eff}(x) = Ip_{eff}$ is a constant in our case.

$$\varphi_c = \int_0^{2l} \frac{d\varphi}{dx} dx = \int_0^{l} \frac{-2hw}{G * Ip_{eff}} dx + \int_l^{2l} \frac{-hw}{G * Ip_{eff}} dx = \frac{-3lhw}{G * Ip_eff}$$

$$\varphi_c = \frac{-3lhw}{G * Ip_eff} = \frac{-0.6m * 15Nm}{26MPa * \pi/2 * (0.0075m)^4} = -0.07rad$$

$$\varphi_c h = -0.07rad * 5cm = -0.35cm$$

4.3.2 Calculating v_c

To calculate v_c we will take advantage of the Myosotis Tables. See the Myosotis table below of a beam or shaft supported on two ends.

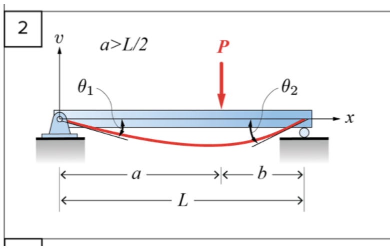
	$v(x) = \frac{-Pbx}{6EIL}(L^2 - b^2 - x^2)$ $0 \le x \le a$ $v(x) = \frac{-Pb}{6LEI}\left[\frac{L}{b}(x-a)^3 - x^3 + (L^2 - b^2)x\right]$ $a \le x \le L$	$\theta_1 = \frac{-Pab(L+b)}{6EIL}$ $\theta_2 = \frac{Pab(L+a)}{6EIL}$	$v_{max} = \frac{-Pb(L^2-b^2)^{\frac{3}{2}}}{9\sqrt{3}LEI}$ $at\ x = \sqrt{\frac{L^2-b^2}{3}}$

Figure 12: Myosotis Table for the deflection of a beam or shaft relevant to this problem.

For our system we will use the superposition of two loads $P = -w$ at $a = 2l$ and $a = l$, $L = 3l$ in this case. Note that both forces F_1 and F_2 will provide a contribution to v_c and the total v_c is the sum of the individual contributions from each force calculated using the table.

Also note that due to symmetry the deflection at l in the F_2 scenario is equal to the deflection at $2l$ for the F_1 scenario! This might make thinking about the problem a little easier!

$$F_2 \ \rightarrow \ v_c = \frac{2wl}{18EI}[9l^2 - l^2 - 4l^2] = \frac{4wl^3}{9EI}$$

$$F_1 \ \rightarrow \ v_c = \frac{wl}{18EI}[9l^2 - l^2 - l^2] = \frac{7wl^3}{18EI}$$

$$v_c = \frac{5wl^3}{6EI} = \frac{5 * 300N * (0.2m)^3}{6 * (70GPa) * (\pi/4 * (0.0075m)^4)} = 1.15cm$$

4.3.3 Putting it all Together

Now we can finally say that the vertical displacement of the rope v_d is as follows.

$$v_d = v_c - \varphi_c h = 1.15cm + 0.35cm = 1.5cm$$

4.4 Part IV: Intuition Building

The last part of the question asked us to use our intuition and not calculations to determine the most failure-vulnerable location in the bar in terms of stress. To figure this out we want to first look at where the shear stress $\tau(x, r)$ is at a max and the axial stress $\sigma(x, r)$ is at a max. We already know that the maximum stresses occur at $r = R$ for these stresses.

$$\tau(x, R) = \begin{cases} -4.53 * 10^7 & x \le l \\ -2.26 * 10^7 & l < x \le 2l \\ 0 & 2l < x \le 3l \end{cases}$$

And the maximum axial stress from bending is in the region $l \le x \le 2l$ where:

$$|\sigma(x)|_{max} = \frac{4 * w * l}{\pi * R^3} = 181.08MPa$$

So we just need to figure out where these *both* are at a maximum and that is at $x = l$ and at the outer radius $r = R$. This is where the shaft is most likely to fail.

5 Analysis

And we're done! That wasn't so bad right? The last thing we want to do is talk about how our assumptions may have affected our outcomes and how we might have been able to model this problem a little better. Let's list the assumptions again.

5.1 Assumptions Re-Evaluated

We now re-list our assumptions but this time we describe why they may be correct or incorrect:

- **General:** We will assume for simplicity sake that the shaft is stationary at the moment of analysis and is not rotating. We will also assume that the torque provided by the motor is perfectly balanced by the torques provided by the ropes.

If we assume the shaft is spinning at constant velocity, the torque will exactly balance the forces from the ropes. The problem is the shaft likely is not spinning at constant velocity and there is some net-torque on the shaft that varies with time as the motor speed varies meaning the loading of the shaft is constantly changing. It's also important to note that if the shaft is spinning the shaft's loading is actually cyclic (the part of the bar feeling the stress is changing even though to our eyes the location of these stresses won't change). The stresses on a part of the shaft rise and fall as it rotates so fatigue may be an issue here. These are things we didn't account for in our calculation.

- **Zeros and Origin:** We will assume for simplicity sake that $\varphi(x = 0) = 0$ and that the displacements $u_A = 0$ and $u_B = 0$ where A and B are the locations of the bearings on the shaft. The location of $x = 0$ is at point A.

Again, if the shaft is spinning, technically $\varphi(x = 0) \neq 0$ in our stationary reference frame. However we can say that if we change our reference frame to the rotating frame of the shaft, if we wanted to calculate the angle of rotation of the shaft we could simply calculate $\varphi_B - \varphi_A$ and say that $\varphi_A = 0$. This did not affect any calculations, however. We *can* say that $u_A = 0$ and $u_B = 0$ for the shaft assuming the bearings are doing their job but we assumed that the angle of deflection of the beam at A and B (θ_A and θ_B) are non-zero which is not strictly correct. If the bearings are large enough (which they likely are) they could act like walls making $\theta_A = \theta_B = 0$. Depending on the size of the bearings this assumption may have to be re-visited.

- **Loading Conditions:** We will also assume that the ropes are evenly spaced across the shaft and that they are being wrapped around pullies with a radius of 5cm. We will assume the bearings can only provide forces in the y and z direction and the system has no forces on it in the x direction as the ropes pull perfectly in the vertical direction.

Realistically, the biggest issue here is the fact that the ropes when wound won't stay at the same x coordinate of the shaft. When a rope winds up it travels back and forth along the spool changing the locations of the force being applied. Another issue is the fact that the assumption that the ropes are only pulling in the vertical direction is questionable. There's likely some angle to them and therefore some x force being applied to the shaft. This is something we didn't account for. This mechanism would likely need retention rings (or similar) on the shaft to stop it from sliding in the x direction.

5.2 A Note From Your Mentors

CONGRATULATIONS YOU DID IT!!!! After Project 3, you probably already know you're officially done with all of your projects for 2.001! It was so much fun being one of your mentors and I hope you've learned as much from me as I from you! Y'all are gonna be some great engineers—proud of you guys! ♡